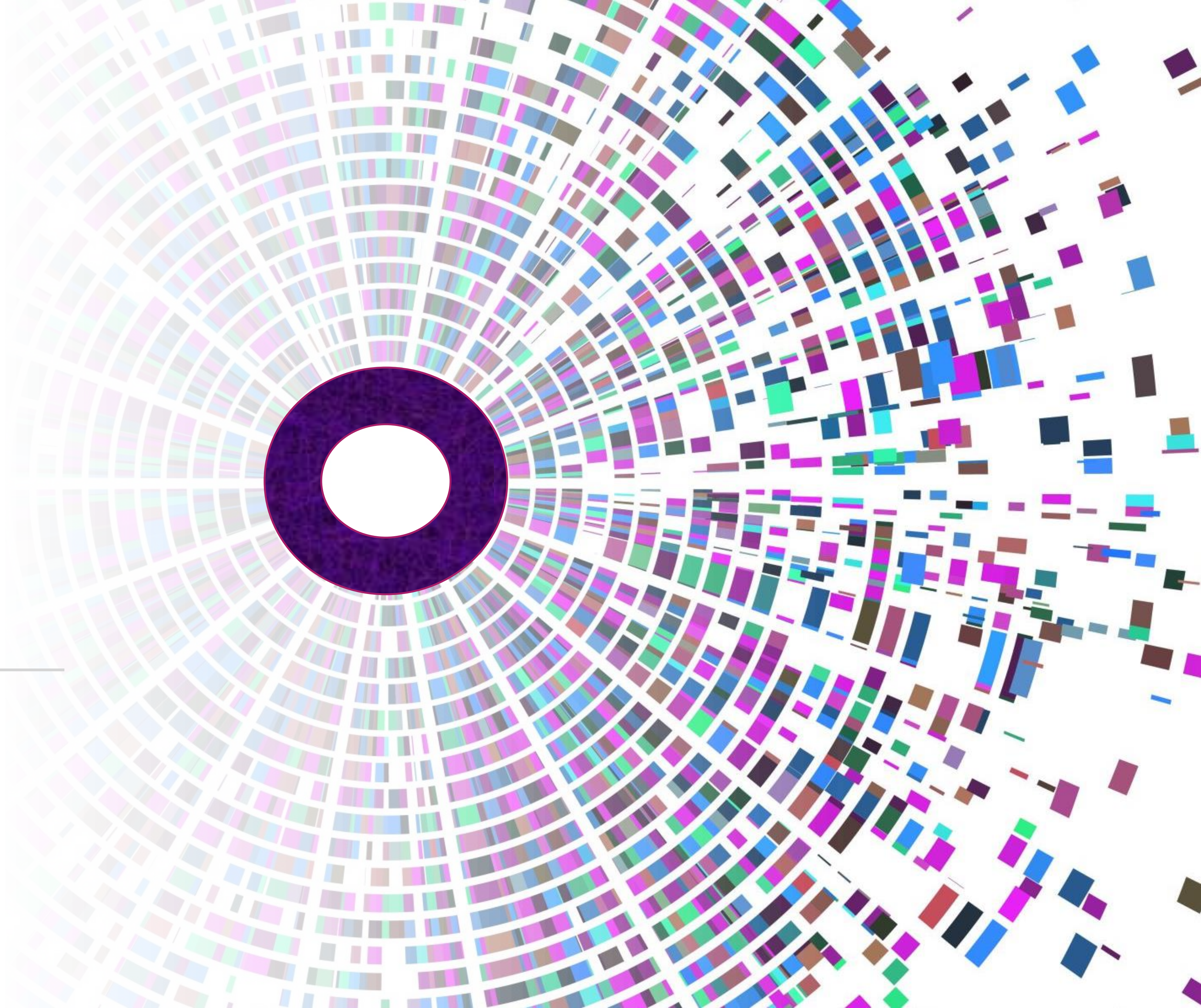
What Is: Furstenberg’s $\times 2, \times 3$ Theorem?



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What is a Torus?

Let V be a vector space and $L \subseteq V$ be a lattice, that is a discrete set of complete dimension. Then, V/L is a torus.

<u>Examples:</u>

- $\mathbb{R}/\mathbb{Z} \cong [0,1)$ and in general $\mathbb{R}/\alpha\mathbb{Z}$, where $0 \neq \alpha \in \mathbb{R}$.
- $\mathbb{R}^n/\mathbb{Z}^n$
- $\mathbb{R}^n/X\mathbb{Z}^n$, where $X \in GL_n(\mathbb{R})$
- $\mathbb{F}_p((x^{-1}))/\mathbb{F}_p[x]$

Question

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong [0,1)$ be the one dimensional torus. Which sets are invariant under the maps $\times\, n$ when n is a natural number? Meaning which subsets of the torus satisfy

$$\{na\ (mod\ 1): a \in A\} = nA = A$$

Clearly

$$\mathrm{A} = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots \frac{n-2}{n-1}\right\}$$

is $\times\, n$ invariant.

Are there any other $\times\, n$ invariant sets? Can we construct such a set that is infinite?

Answer for $n = 2$ and Motivation

Since $\mathbb{T} \cong [0,1)$, each $x \in \mathbb{T}$ has a binary expansion

$$x = \sum_{n=1}^{\infty} a_n 2^{-n}$$

where $a_n \in \{0,1\}$. Write $x = (a_1, a_2, \ldots)$. Then,

$$2x \ mod\ 1 = \sum_{n=1}^{\infty} a_{n+1} 2^{-n} = (a_2, a_3, \ldots)$$

Answer for $n = 2$ and Motivation

Denote $\Omega = \{0,1\}^{\mathbb{N}}$ to be the set of binary sequences, and σ, the shift map,

$$\sigma(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$$

Then, $A \subseteq \mathbb{T}$ is $\times\, 2$ invariant $\Leftrightarrow$ the set of binary sequences corresponding to numbers in A is shift invariant.

From here we can construct such a set

$$A = \left\{ \sum_{n=1}^{\infty} a_n 2^{-n} : a_{2n} = 0\ for\ each\ n, or\ a_{2n-1} = 0\ for\ each\ n \right\}$$

Question

Can we construct infinite sets that are invariant under both $\times 2, \times 3$ that are not $\mathbb{T}$?

<u>Theorem (Furstenberg 1967):</u>

If $n^k \neq m^l$ for each $k, l \in \mathbb{Z}$, then the only infinite closed subset of $\mathbb{T}$ invariant under both $\times n$ and $\times m$ is $\mathbb{T}$ itself.

In general, if Δ is a multiplicative semigroup that is not of the form $\{a^n : n \in \mathbb{N}\}$, then the conclusion holds.

Consequence of Furstenberg's Theorem

Let $\alpha \in \mathbb{T}$. The set

$$\mathcal{O}_\Delta(\alpha) = \{n^k m^l \alpha : k, l \in \mathbb{N}\}$$

is the orbit of α under the semigroup

$$\Delta = \{n^k m^l : k, l \in \mathbb{N}\}$$

Then, $\overline{\mathcal{O}_\Delta(\alpha)}$ is a closed set invariant under $\times n, \times m$.

Therefore, if it is infinite, then it is $\mathbb{T}$.

Thus, each orbit of $\times n, \times m$ is either finite or dense.

We say that a semigroup Δ is ID (infinite⇒dense) if each of its orbits are either finite or dense.

Application To Diophantine Approximation

Let $\Delta \subseteq \mathbb{N}$. Can we find for each $\varepsilon > 0$ and irrational α, some $n \in \Delta$ and $m \in \mathbb{Z}$ such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{\varepsilon}{n}$$

<u>Hardy-Littlewood (1914):</u> In order for this to hold, Δ must satisfy for each $\alpha \notin \mathbb{Q}$,

$$\{n\alpha \ mod\ 1 : n \in \Delta\}$$

must be dense in $\mathbb{T}$.

Application To Diophantine Approximation

By Furstenberg's theorem, for $\Delta = \{2^k 3^l : k, l \in \mathbb{N}\}$ all orbits are either dense or finite. Let α be irrational. Then,

$$\{2^k \alpha \; mod \; 1 : k \in \mathbb{N}\}$$

is infinite, since if not, then,

$$\{2^k \alpha \; mod \; 1 : k \in \mathbb{N}\} = \{2\alpha, 4\alpha, \ldots 2^n \alpha\}$$

Then, $2^{n+1}\alpha = 2^j \alpha \; mod \; 1$ for $j \leq n$.

Thus, $\alpha(2^{n+1-j} - 1)$ would be an integer, so that α would be rational, contradiction!

Application To Diophantine Approximation

Therefore, the set $\{2^n 3^m \alpha : n, m \in \mathbb{N}\}$ is infinite and also $\times\, 2$ and $\times\, 3$ invariant.

Hence, by Furstenberg's theorem,

$$\{2^n 3^m \alpha : n, m \in \mathbb{N}\}$$

is dense.

Thus, by Hardy-Littlewood, for each irrational α and $\varepsilon > 0$, exist

$n, m \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$\left|\alpha - \frac{k}{2^n 3^m}\right| < \frac{\varepsilon}{2^n 3^m}$$

Generalizations and Analogs

- Can we generalize such a theorem to multidimensional tori

  $\mathbb{T}^r = \mathbb{R}^r/\mathbb{Z}^r$?

  Yes! Berend proved such a result.

- Do such analogs exist for tori over fields of positive characteristic? What does this even mean?

Multi-Dimensional Generalization

Theorem (Berend 1983):

Let $\mathbb{T}^r = \mathbb{R}^r/\mathbb{Z}^r$ be the r dimensional torus.

Let $\Sigma \subseteq M_r(\mathbb{Z})$ be a commutative semigroup acting on the torus by $A \cdot x = Ax$. Then, Σ is ID if and only if the following conditions are satisfied:

1. Σ contains a pair of matrices σ, τ such that $\sigma^m \neq \tau^n$ for all $n, m \in \mathbb{Z}$
2. Exists $\sigma \in \Sigma$ such that the characteristic polynomial of σ^n is irreducible over $\mathbb{Z}$ for each $n \in \mathbb{N}$.
3. For every v which is an eigenvector of each $\sigma \in \Sigma$, there exists a matrix $\sigma \in \Sigma$, such that $\sigma v = \lambda v$ and $|\lambda| > 1$.

Necessity of the Conditions

- Why must the characteristic polynomial of some σ^n be irreducible over $\mathbb{Z}$ for all $n \in \mathbb{N}$?

  Take Σ to be generated by
  $$A = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix}$$
  Then the set of vectors $(v_1, v_2) \in \mathbb{T}^2$ where for $i = 1,2$ the binary expansion of v_i contains only even or only odd powers. Then, this set is invariant under
  $$\{A^n B^m : n, m \in \mathbb{N}\}$$

Generalizations and Analogs

- Can we generalize such a theorem to multidimensional torii
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  Yes! Berend proved such a result.
- Do such analogs exist for tori over fields of positive characteristic? What does this even mean?

Positive Characteristic Setting

Structure	Characteristic 0	Characteristic p
Discrete Ring	$\mathbb{Z}$	$\mathbb{F}_p[x]$
Field of Fractions	$\mathbb{Q}$	$\mathbb{F}_p(x)$
Topological Closure	$\mathbb{R}$	$\mathbb{F}_p\left((x^{-1})\right)$
Algebraic Closure	$\mathbb{C}$	$\overline{\mathbb{F}_p\left((x^{-1})\right)}$
Torus	$\mathbb{R}/\mathbb{Z}$	$\mathbb{F}_p((x^{-1}))/\mathbb{F}_p[x]$

Positive Characteristic Analogs

<u>Setting:</u>

Let p be prime and let $\mathbb{F}_p\left((x^{-1})\right)$ be the field of Laurent polynomials of the form

$$\mathbb{F}_p\left((x^{-1})\right) = \left\{ \sum_{n=-\infty}^{N} a_n x^n : N \in \mathbb{Z}, a_n \in \mathbb{F}_p \right\}$$

Then, a torus would be

$$\mathbb{T}_p = \mathbb{F}_p\left((x^{-1})\right) / \mathbb{F}_p[x] = \left\{ \sum_{n=1}^{\infty} a_n x^{-n} : a_n \in \mathbb{F}_p \right\}$$

Positive Characteristic Analogs

Furstenberg's theorem fails in positive characteristic!

<u>Counterexample (Berend 1989):</u>

Take $p = 2$. Take $f(x) = x$ and $g(x) = x + 1$. Then, the polynomials f, g are coprime. But take

$$B = \left\{ \sum_{n=1}^{\infty} a_n x^{-2n} : n \in \mathbb{N} \right\}$$

Then,

$$f(x)B = \left\{ \sum_{n=1}^{\infty} a_n x^{-(2n-1)} : n \in \mathbb{N} \right\}$$

Positive Characteristic Analogs

Also, $g(x)B$ is the set of sequences $(a_1, a_1, a_2, a_2, \dots)$ since

$$(x+1)\sum_{n=1}^{\infty} a_n x^{-2n} = \sum_{n=1}^{\infty} a_n x^{-(2n-1)} + \sum_{n=1}^{\infty} a_n x^{-2n}$$

Similarly, $g(x)f(x)B$ is the set of sequences $(a_1, a_2, a_2, a_3, a_3, \dots)$:

$$(x+1)\sum_{n=1}^{\infty} a_n x^{-(2n-1)} = \sum_{n=1}^{\infty} a_n x^{-(2n-2)} + \sum_{n=1}^{\infty} a_n x^{-2n}$$

Positive Characteristic Analogs

For a sequence in $g(x)B$,

$$(x+1)\left(\sum_{n=1}^{\infty} a_n x^{-2n} + \sum_{n=1}^{\infty} a_n x^{-(2n-1)}\right)$$
$$= 2\sum_{n=1}^{\infty} a_n\, x^{-(2n-1)} + \sum_{n=1}^{\infty} a_n x^{-2n} + \sum_{n=1}^{\infty} a_n x^{-(2n-2)}$$

So that $g^2(x)B = B$. Similarly, $g^2(x)f(x)B = f(x)B$.

Positive Characteristic Analogs

Take

$$A = B \cup f(x)B \cup g(x)B \cup f(x)g(x)B$$

Then, A is clearly infinite, invariant and closed. But A is not $\mathbb{T}_p$, since

$$x^{-2} + x^{-3} + x^{-5} \notin A$$

Since it has both even and odd powers as well as consecutive pairs that are not equal one to another.

Thus, Furstenberg's theorem fails in positive characteristic!

Finishing Remarks and Questions

- Can we strengthen the conditions in Furstenberg's theorem so that the conclusion can be true in $\mathbb{F}_p\left((x^{-1})\right)$? What is the correct notion of large?
- Is an analog of Berend's multi-dimensional theorem true in the positive characteristic setting?
- What measures are invariant under both $\times 2, \times 3$? What can be said about measure rigidity?
- What if we take non-abelian semigroups?
- Are there other applications of Furstenberg's and Berend's theorems?

Any Questions?

Thank you for listening

References

- Hillel Furstenberg. Disjointness in Ergodic Theory, Minimal Sets, and a Problem in Diophantine Approximation. *Mathematical systems theory,* 1:1-49, 1967
- Daniel Berend. Multi-Invariant Sets on Tori. *Transactions of the American Mathematical Society*, Volume 280, 1983
- Daniel Berend. Density Modulo 1 in Function Fields, *Acta Arithmetica,* LII, 1989
- G.H. Hardy and J.E. Littlewood. Some Problems in Diophantine Approximation: Part I. The Fractional Part of $n_k\theta$. *Acta Math*, volume 37, 155-191, 1914

More Reading for the Motivated Student

- A book on Ergodic Theory, which speaks about the dynamics of the shift on a space $A^{\mathbb{N}}$ and the space of lattices among many other things - Manfred Einsiedler, Thomas Ward, Ergodic Theory with a View Towards Number Theory
- An internet book about Homogeneous Dynamics, which goes more into lattices – Manfred Einsiedler, Thomas Ward, Homogeneous Dynamics
- A book on Ergodic Theory, discussing many topics including invariant measures for the shift among other things – Hillel Furstenberg, Recurrence in Ergodic Theory and Combinatorial Number Theory

Reading Related to the Finishing Questions

- Application of Berend's Theorem to Homogeneous Dynamics – U. Shapira. On a Generalization of Littlewood's Conjecture, 2008
- One of many papers on the Measure Rigidity Problem – D. Rudolph. $\times 2, \times 3$ Invariant Measures and Entropy, 1990
- On Topological Rigidity of non-Abelian Actions – Y. Guivarc'h and A.N. Starkov. Orbits of Linear Group Actions, Random Walks on Homogeneous Spaces and Toral Automorphisms
- On Measure Rigidity for non-Abelian Groups – J. Bourgain, A. Furman, E. Lindenstrauss, S. Mozes. Invariant Measures and Stiffness for Non-Abelian Groups of Toral Automorphisms